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# Dipole formalism with heavy fermions

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Abstract: We extend the dipole formalism of Catani and Seymour to QCD processes involving heavy fermions. We give the appropriate subtraction terms together with their integrated counterpart. All calculations are done within dimensional regularization. Our formulae can be used with three variants of dimensional regularization (conventional dimensional regularization, the 't Hooft-Veltman scheme or the four-dimensional scheme).

Keywords: QCD, NLO Computations, Jets.

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## Contents





## 1. Introduction

Studies of hard-scattering processes in QCD with heavy particles are important at present or future colliders. Reliable theoretical prediction require at least the evaluation of next-to-leading order (NLO) QCD corrections. Next-to-leading order calculations have to combine virtual one-loop corrections with the real emission contributions from unresolved partons. Treated separately, each of the two parts gives an infrared divergent contribution. Only the sum of the virtual corrections and the real emission contributions is infrared finite. Setting up a numerical general-purpose NLO Monte-Carlo program requires therefore the analytical cancellation of infrared singularities before any numerical integration can be done. The two main methods to handle this task are phase space slicing [1]–[3] and the subtraction method [4, 5]. For massless partons both methods are available and have been applied to a variety of specific processes, see for example ref. [6]–[10] and the references therein. For electron-positron annihilation a general formulation of phase space slicing has been given by Giele and Glover [11]. It has been extended to initial-state partons by these authors and Kosower [12]. The extension of phase space slicing to massive partons and identified hadrons has been given by Laenen and Keller [13].

There are two general formulations of the subtraction method. One is the residue approach by Frixione, Kunzst and Signer [14] and the other the dipole formalism by Catani and Seymour [15], both variants can handle massless partons in the final and/or initial state. The subtraction method has already been applied to some specific processes with massive partons [16]–[19]. Up to now there is no extension of the dipole formalism to massive partons. Dittmaier has considered the dipole formalism for photon radiation from fermions [20]. In that work infrared divergences are regularized with small masses (as it is popular in electroweak physics). However this does not allow a simple application to QCD, where divergences are usually regularized by dimensional regularization.

In this paper we extend the dipole formalism to heavy fermions. The formulae we provide are relevant to top, bottom and charm production. With a simple change of the colour factors they can be used as well for gluino production in SUSY QCD. Our results are not applicable for processes with identified hadrons in the final state, massive initial-state partons and processes with different species of massive fermions of unequal masses.

One-loop amplitudes may be calculated in different variants of dimensional regularization, such as conventional dimensional regularization (CDR), the 't Hooft — Veltman scheme (HV) or the four-dimensional scheme (FD), whereas the Born amplitudes entering the real emission part and/or parton densities are given in another variant. Of course, the final result has to be scheme-independent. The obvious way to ensure this is to calculate everything in the same scheme. Often this is not the most economical solution. Loop amplitudes are most easily computed in the FD scheme, whereas parton densities are given in the conventional scheme. Within the dipole formalism it is possible to perform different parts of the calculation with different variants of dimensional regularization and to correct for the mismatch by some universal terms [21]. We would like to keep this freedom also in the extension to massive partons. We therefore keep track of all scheme-dependent terms and our final results can be used with any variant of dimensional regularization (CDR, HV or FD).

Recently, Catani, Dittmaier and Trocsanyi considered the singular behaviour of one-loop QCD and supersymmetric QCD amplitudes with massive partons in the dipole framework [22]. As a byproduct, we confirm their results for the singular terms.

This paper is organized as follows. In the next section we review the dipole formalism and the factorization of QCD amplitudes in the soft and collinear limit. In section 3 we outline our calculational technique. In section 4 we derive the $D$-

dimensional dipole phase space measure. Section 5 gives the dipole terms together with the integrated counterpart if all particles are in the final state. In section 6 we consider the case if there are QCD partons in the initial state. Finally, section 7 contains our conclusions.

## 2. The dipole formalism and factorization in singular limits

In this section we briefly review the dipole formalism and the factorization properties of QCD amplitudes in the soft and collinear limit. We use the notation of Catani and Seymour [15].

The dipole formalism is based on the subtraction method. We explain it for electron-positron annihilation, where all QCD partons are in the final state. The NLO cross section is rewritten as

$$\begin{aligned}\sigma^{\mathrm{NLO}} &= \int_{n+1} d\sigma^R + \int_n d\sigma^V \\ &= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right). \qquad (2.1)\end{aligned}$$

In the second line an approximation term $d\sigma^A$ has been added and subtracted. The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour (in $D$ dimensions) as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$. This defines a cross-section contribution

$$\sigma^{\mathrm{NLO}}_{\{n+1\}} = \int_{n+1} \left( d\sigma^R\big|_{\varepsilon=0} - d\sigma^A\big|_{\varepsilon=0}\right). \qquad (2.2)$$

- Analytic integrability (in $D$ dimensions) over the one-parton subspace leading to soft and collinear divergences. This gives the contribution

$$\sigma^{\mathrm{NLO}}_{\{n\}} = \int_n \left( d\sigma^V + \int_1 d\sigma^A \right)_{\varepsilon=0}. \qquad (2.3)$$

The final structure of an NLO calculation is then

$$\sigma^{\mathrm{NLO}} = \sigma^{\mathrm{NLO}}_{\{n+1\}} + \sigma^{\mathrm{NLO}}_{\{n\}}. \qquad (2.4)$$

Since both contributions on the r.h.s. of eq. (2.4) are now finite, they can be evaluated with numerical methods. The $(n+1)$ matrix element is approximated by a sum of dipole terms

$$\begin{aligned}&\sum_{\text{pairs } i,j} \sum_{k \neq i,j} \mathcal{D}_{ij,k} = \qquad (2.5)\\ &= \sum_{\text{pairs } i,j} \sum_{k \neq i,j} -\frac{1}{2 p_i \cdot p_j} \langle 1, \ldots, \tilde{(ij)}, \ldots, \tilde{k}, \ldots | \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} V_{ij,k} | 1, \ldots, \tilde{(ij)}, \ldots, \tilde{k}, \ldots \rangle,\end{aligned}$$

where the emitter parton is denoted by $\tilde{ij}$ and the spectator by $\tilde{k}$. Here $\mathbf{T}_i$ denotes the colour charge operator [15] for parton $i$ and $V_{ij,k}$ is a matrix in the helicity space of the emitter with the correct soft and collinear behaviour. $|1, \ldots, (\tilde{ij}), \ldots, \tilde{k}, \ldots\rangle$ is a vector in colour- and helicity space. By subtracting from the real emission part the fake contribution we obtain

$$d\sigma^R - d\sigma^A = d\phi_{n+1}\Bigg[|M(p_1, \ldots, p_{n+1})|^2\theta^{\rm cut}_{n+1}(p_1, \ldots, p_{n+1}) - \\ - \sum_{\text{pairs } i,j}\sum_{k\neq i,j}\mathcal{D}_{ij,k}(p_1, \ldots, p_{n+1})\theta^{\rm cut}_n(p_1, \ldots, \tilde{p}_{ij}, \ldots, \tilde{p}_k, \ldots, p_{n+1})\Bigg]. \tag{2.6}$$

Both $d\sigma^R$ and $d\sigma^A$ are integrated over the same $(n+1)$ parton phase space, but it should be noted that $d\sigma^R$ is proportional to $\theta^{\rm cut}_{n+1}$, whereas $d\sigma^A$ is proportional to $\theta^{\rm cut}_n$. Here $\theta^{\rm cut}_n$ denotes the jet-defining function for $n$-partons.

The subtraction term can be integrated over the one-parton phase space to yield the term

$$\mathbf{I}\otimes d\sigma^B = \int_1 d\sigma^A = \sum_{\text{pairs } i,j}\sum_{k\neq i,j}\int d\phi_{\rm dipole}\mathcal{D}_{ij,k}\,. \tag{2.7}$$

The universal factor $\mathbf{I}$ still contains colour correlations, but does not depend on the unresolved parton $j$. The term $\mathbf{I}\otimes d\sigma^B$ lives on the phase space of the $n$-parton configuration and has the appropriate singularity structure to cancel the infrared divergences coming from the one-loop amplitude. Therefore

$$d\sigma^V + \mathbf{I}\otimes d\sigma^B \tag{2.8}$$

is infrared finite and can easily be integrated by Monte Carlo methods.

In order to extend the dipole formalism to massive fermions, we have to provide three ingredients. First, we need the correct subtraction terms for the real emission contribution. Second, we have to integrate these subtraction terms over the dipole phase space. The integrated terms are combined with the virtual corrections. Third, we have to specify a mapping of the momenta, which relates the $(n+1)$-parton configuration $p_1, \ldots, p_i, \ldots, p_j, \ldots p_k, \ldots p_{n+1}$ to the $n$-parton configuration $p_1$, $\ldots$, $\tilde{p}_{ij}, \ldots, \tilde{p}_k, \ldots, p_{n+1}$.

In order to find appropriate dipole terms, one considers the soft and collinear limits of the matrix element. In the soft limit where the momentum of parton $j$ becomes soft, the $m+1$-parton matrix element behaves as

$$\langle 1, \ldots, m+1|1, \ldots, m+1\rangle = -4\pi\mu^{2\varepsilon}\alpha_s\langle 1, \ldots, m|\mathbf{J}^{\mu\dagger}\mathbf{J}_\mu|1, \ldots, m\rangle\,, \tag{2.9}$$

with

$$\mathbf{J}^{\mu\dagger}\mathbf{J}_\mu|1, \ldots, m\rangle = \left(2\sum_{i\neq k}\frac{2p_ip_k}{(2p_ip_j)(2p_jp_k)}\mathbf{T}_i\cdot\mathbf{T}_k + \sum_i\frac{4m_i^2}{(2p_ip_j)^2}\mathbf{T}_i\cdot\mathbf{T}_i\right)|1, \ldots, m\rangle \\ = 4\sum_{i\neq k}\left(\frac{2p_ip_k}{(2p_ip_j)(2p_ip_j+2p_jp_k)} - \frac{m_i^2}{(2p_ip_j)^2}\right)\mathbf{T}_i\cdot\mathbf{T}_k|1, \ldots, m\rangle\,. \tag{2.10}$$

We used

$$\frac{2p_ip_k}{(2p_ip_j)(2p_jp_k)} = \frac{2p_ip_k}{(2p_ip_j)(2p_ip_j+2p_jp_k)} + \frac{2p_ip_k}{(2p_jp_k)(2p_ip_j+2p_jp_k)} \tag{2.11}$$

and colour conservation

$$\sum_i \mathbf{T}_i|1,\ldots,m\rangle = 0\,. \tag{2.12}$$

The colour charge operators $\mathbf{T}_i$ for a quark, gluon and antiquark in the final state are

$$\begin{aligned} \text{quark}: &\ \langle\ldots q_i\ldots|T^a_{ij}|\ldots q_j\ldots\rangle\,,\\ \text{gluon}: &\ \langle\ldots g^c\ldots|if^{cab}|\ldots g^b\ldots\rangle\,,\\ \text{antiquark}: &\ \langle\ldots \bar{q}_i\ldots|\left(-T^a_{ji}\right)|\ldots \bar{q}_j\ldots\rangle\,. \end{aligned} \tag{2.13}$$

If the particles are massless, there is also a singularity in the collinear limit. For final-state partons the momenta are parameterized as

$$\begin{aligned} p_i &= \hat{z}p + k_\perp - \frac{k_\perp^2}{\hat{z}}\frac{n}{2pn}\,,\\ p_j &= (1-\hat{z})p - k_\perp - \frac{k_\perp^2}{1-\hat{z}}\frac{n}{2pn}\,. \end{aligned} \tag{2.14}$$

Here $n$ is a massless four-vector and the transverse component $k_\perp$ satisfies $2pk_\perp = 2nk_\perp = 0$. The collinear limits occurs for $k_\perp^2 \to 0$. In this limit the matrix element behaves as

$$\langle 1,\ldots,m+1|1,\ldots,m+1\rangle = 4\pi\mu^{2\varepsilon}\alpha_s\langle 1,\ldots,m|\frac{2}{2p_ip_j}\hat{P}_{(ij),i}(\hat{z},k_\perp,\varepsilon)|1,\ldots,m\rangle\,. \tag{2.15}$$

The splitting function are given by

$$\begin{aligned} \langle s|\hat{P}_{qq}(z,k_\perp,\varepsilon)|s'\rangle &= \delta_{ss'}C_F\left[\frac{2z}{1-z}+(1-\rho\varepsilon)(1-z)\right],\\ \langle s|\hat{P}_{qg}(z,k_\perp,\varepsilon)|s'\rangle &= \delta_{ss'}C_F\left[\frac{2(1-z)}{z}+(1-\rho\varepsilon)z\right],\\ \langle \mu|\hat{P}_{gq}(z,k_\perp,\varepsilon)|\nu\rangle &= T_R\left[-g^{\mu\nu}+4z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right],\\ \langle \mu|\hat{P}_{gg}(z,k_\perp,\varepsilon)|\nu\rangle &= 2C_A\left[-g^{\mu\nu}\left(\frac{z}{1-z}+\frac{1-z}{z}\right)-2(1-\rho\varepsilon)z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right]. \end{aligned} \tag{2.16}$$

We introduced the parameter $\rho$, which specifies the variant of dimensional regularization: $\rho = 1$ for the CDR/HV schemes and $\rho = 0$ for the FD scheme. Later on we will chose the dipole terms to have the same soft and collinear behaviour as the appropriate limit of the $(n+1)$-parton matrix element.



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If the emitting particle is in the initial state the collinear limit is defined as

$$\begin{aligned} p_a &= p\,, \\ p_i &= (1-x)p + k_\perp - \frac{k_\perp^2}{1-x}\frac{n}{2pn}\,, \\ p_{ai} &= xp - k_\perp + \frac{k_\perp^2}{1-x}\frac{n}{2pn}\,. \end{aligned} \tag{2.17}$$

The colour charge operators for a quark, gluon and antiquark in the initial state are

$$\begin{aligned} \text{quark}:&\ \langle\ldots\bar{q}_i\ldots|\left(-T^a_{ji}\right)|\ldots\bar{q}_j\ldots\rangle\,, \\ \text{gluon}:&\ \langle\ldots g^c\ldots|if^{cab}|\ldots g^b\ldots\rangle\,, \\ \text{antiquark}:&\ \langle\ldots q_i\ldots|T^a_{ij}|\ldots q_j\ldots\rangle\,. \end{aligned} \tag{2.18}$$

We denoted in the amplitude an incoming quark as an outgoing antiquark and vice versa.

## 3. Calculational technique for the integration

The integration of the dipole terms over the dipole phase space is highly non-trivial. We first find a suitable parameterization of the phase space such that all integrals are of one of the following types:

$$\begin{aligned} \int_0^1 dx x^{a-1}(1-x)^{c-a-1} &= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\,, \\ \int_0^1 dx x^{a-1}(1-x)^{c-a-1}(1-x_0x)^{-b} &= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}{}_2F_1(a,b;c,x_0)\,, \\ \int_0^1 dx x^{a-1}(1-x)^{c-a-1}(1-x_1x)^{-b_1}(1-x_2x)^{-b_2} &= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F_1(a,b_1,b_2;c;x_1,x_2)\,. \end{aligned} \tag{3.1}$$

These integrals yield the Euler-Beta function, the hypergeometric function and the first Appell function, respectively. The last two are then rewritten as a Taylor series in $x_0$ or $x_1$ and $x_2$, respectively.

$$\begin{aligned} {}_2F_1(a,b;c,x_0) &= \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{x_0^n}{n!}\,, \\ F_1(a,b_1,b_2;c;x_1,x_2) &= \sum_{m_1=0}^\infty\sum_{m_2=0}^\infty \frac{(a)_{m_1+m_2}(b_1)_{m_1}(b_2)_{m_2}}{(c)_{m_1+m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\,. \end{aligned} \tag{3.2}$$

Here $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol. Taylor expansion allows us to perform the next integral, which is of the same type as above. After all integrals have been performed, we end up with multiple sums involving $\Gamma$-functions depending on $\varepsilon = (4-D)/2$. These $\Gamma$-functions are then expanded according to

$$\frac{\Gamma(n+\varepsilon)}{\Gamma(n)\Gamma(\varepsilon)} = \varepsilon\Big(1 + \varepsilon Z_1(n-1) + \varepsilon^2 Z_{11}(n-1) + \varepsilon^3 Z_{111}(n-1) + \cdots \\ \cdots + \varepsilon^{n-1} Z_{11\ldots1}(n-1)\Big), \tag{3.3}$$

where $Z_{11\ldots1}(n)$ are Euler-Zagier sums defined by

$$Z_{m_1,\ldots,m_k}(n) = \sum_{i_1>i_2>\cdots>i_k>0}^{n} \frac{1}{i_1^{m_1}} \frac{1}{i_2^{m_2}} \cdots \frac{1}{i_k^{m_k}}\,. \tag{3.4}$$

Rearranging summation indices we recognize that all sums fall in the class of Goncharov's multiple polylogarithms [23, 24]

$$\mathrm{Li}_{m_k,\ldots,m_2,m_1}(x_k,\ldots,x_2,x_1) = \sum_{i_1>i_2>\cdots>i_k>0}^{\infty} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}\,. \tag{3.5}$$

In the case with only one massive parton it is actually sufficient to restrict oneself to harmonic polylogarithms [25, 26], defined by

$$H_{m_1,m_2\ldots,m_k}(x) = \sum_{i_1>i_2>\cdots>i_k>0}^{\infty} \frac{x^{i_1}}{i_1^{m_1}} \frac{1}{i_2^{m_2}} \cdots \frac{1}{i_k^{m_k}}\,. \tag{3.6}$$

The additional dipole corresponding to gluon emission from a massive quark-antiquark system can be expressed in terms of two-dimensional harmonic polylogarithms [27].

## 4. The dipole phase space measure in D dimensions

In this section we derive the appropriate phase space measure for the dipoles with massive particles. Since singularities are regulated by dimensional regularization, this has to be done in $D$ dimensions. The phase space measure for $n$ particles in $D$ dimensions is given by [28]

$$\begin{aligned} d\phi_n(P \to p_1,\ldots,p_n) &= (2\pi)^D \delta^D\left(P - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{d^D p_i}{(2\pi)^{D-1}} \theta(p_i^0)\delta(p_i^2 - m_i^2) \\ &= (2\pi)^D \delta^D\left(P - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{d^{D-1} p_i}{(2\pi)^{D-1} 2E_i} \end{aligned} \tag{4.1}$$

with

$$E_i = \sqrt{(\vec{p}_i)^2 + m_i^2}\,. \tag{4.2}$$

The phase space measure factorizes according to

$$d\phi_n(P \to p_1,\ldots,p_n) = \frac{1}{2\pi} d\phi_{n-j+1}(P \to Q, p_{j+1},\ldots,p_n) dQ^2 d\phi_j(Q \to p_1,\ldots,p_j)\,. \tag{4.3}$$

### 4.1 Phase space measure with no initial-state particles

We first evaluate the two particle phase space

$$\int d\phi_2(P\to\tilde p_{ij},\tilde p_k) = \int \frac{d^{D-1}\tilde p_{ij}}{(2\pi)^{D-1}2E_{ij}}\frac{d^{D-1}\tilde p_k}{(2\pi)^{D-1}2E_k}(2\pi)^D\delta^D(P-\tilde p_{ij}-\tilde p_k) \tag{4.4}$$

in the rest frame of $P$, e.g. $P=(\sqrt{P^2},\vec 0)$. We obtain

$$\begin{aligned}\int d\phi_2(P\to\tilde p_{ij},\tilde p_k) &= (2\pi)^{2-D}\left(\frac{1}{2}\right)^{D-1}\left(P^2\right)^{1-D/2}\times \\ &\times\left(\sqrt{\left(P^2-m_{ij}^2-m_k^2\right)^2-4m_{ij}^2m_k^2}\right)^{D-3}\int d\Omega_{D-1}\,,\end{aligned} \tag{4.5}$$

where $\Omega_{D-1}$ parameterizes the solid angle of the $(D-1)$ spatial components of $\tilde p_k$ in $(D-1)$ (spatial) dimensions. With this convention we have

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma\left(D/2\right)}\,. \tag{4.6}$$

We then evaluate the three particle phase space

$$\begin{aligned}\int d\phi_3(P\to p_i,p_j,p_k) = \int &\frac{d^{D-1}p_i}{(2\pi)^{D-1}2E_i}\frac{d^{D-1}p_j}{(2\pi)^{D-1}2E_j}\frac{d^{D-1}p_k}{(2\pi)^{D-1}2E_k}\times \\ &\times(2\pi)^D\delta^D(P-p_i-p_j-p_k)\end{aligned} \tag{4.7}$$

in the rest frame of $P$, e.g. $P=(\sqrt{P^2},\vec 0)$. We shall orient our frame such that the solid angle of the spatial components of $p_k$ coincides with the solid angle of the spatial components of $\tilde p_k$ in (4.4). It will be convenient to parameterize the spatial components of $p_i$ with spherical coordinates, using $\vec p_k$ as polar axis, e.g.

$$d^{D-1}p_i = \left|\vec p_i\right|^{D-2}d\left|\vec p_i\right|d\theta_1\sin^{D-3}\theta_1 d\Omega_{D-2}^{(i)}\,. \tag{4.8}$$

We therefore have

$$2\vec p_i\vec p_k = 2\left|\vec p_i\right|\left|\vec p_k\right|\cos\theta_1\,. \tag{4.9}$$

Finally we obtain the three-particle phase space as the product of a two-particle phase space and a dipole phase space:

$$\int d\phi_3(P\to p_i,p_j,p_k) = \int d\phi_2(P\to\tilde p_{ij},\tilde p_k)d\phi_{\rm dipole}(2p_ip_j,2p_jp_k,2p_ip_k,\Omega_{D-2}^{(i)})\,, \tag{4.10}$$

where

$$\begin{aligned}d\phi_{\rm dipole}(2p_ip_j,2p_jp_k,2p_ip_k) &= (2\pi)^{1-D}\frac{2\pi^{\frac{D}{2}-1}}{\Gamma\left(\frac{D}{2}-1\right)}\frac{1}{4}\left(\left(P^2-m_i^2-m_k^2\right)^2-4m_i^2m_k^2\right)^{\frac{3-D}{2}}\times \\ &\times\int d(2p_ip_j)d(2p_jp_k)d(2p_ip_k)\times \\ &\quad\times\delta\left(P^2-m_i^2-m_j^2-m_k^2-2p_ip_j-2p_jp_k-2p_ip_k\right)\times \\ &\quad\times\left(-P^2\lambda\left(|\vec p_i|^2,|\vec p_j|^2,|\vec p_k|^2\right)\right)^{\frac{D-4}{2}}\times \\ &\quad\times\theta\left(-\lambda\left(|\vec p_i|^2,|\vec p_j|^2,|\vec p_k|^2\right)\right)\,.\end{aligned} \tag{4.11}$$

Note that we already performed the angular integration over $d\Omega^{(i)}_{D-2}$. The triangle function $\lambda$ is defined by

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx\,. \tag{4.12}$$

We further have

$$\begin{aligned}
|\vec{p}_i|^2 &= \frac{1}{4P^2}\left(P^2 + m_i^2 - m_k^2 - m_j^2 - 2p_jp_k\right)^2 - m_i^2\,,\\
|\vec{p}_k|^2 &= \frac{1}{4P^2}\left(P^2 - m_i^2 + m_k^2 - m_j^2 - 2p_ip_j\right)^2 - m_k^2\,,\\
|\vec{p}_j|^2 &= \frac{1}{4P^2}\left(P^2 - m_i^2 - m_k^2 + m_j^2 - 2p_ip_k\right)^2 - m_j^2
\end{aligned} \tag{4.13}$$

so that

$$\begin{aligned}
-P^2\lambda\left(|\vec{p}_i|^2, |\vec{p}_j|^2, |\vec{p}_k|^2\right) &= 2p_ip_k 2p_ip_j 2p_jp_k - m_i^2\left(2p_jp_k\right)^2 - m_k^2\left(2p_ip_j\right)^2 - \\
&\quad - m_j^2\left(2p_ip_k\right)^2 + 4m_i^2m_j^2m_k^2\,.
\end{aligned} \tag{4.14}$$

To find a suitable parametrization of the dipole phase space, we treat the various cases separately. We consider first the case where two particles are massless $m_i = m_j = 0$. With $D = 4 - 2\varepsilon$ the dipole phase space becomes

$$\begin{aligned}
d\phi_{\text{dipole}} &= \frac{(4\pi)^{\varepsilon-2}}{\Gamma(1-\varepsilon)}\left(P^2\right)^{1-\varepsilon}(u_0)^{2-2\varepsilon} \times \\
&\quad \times \int_0^1 du u^{1-2\varepsilon}(1-u)^{1-2\varepsilon}\left(1-u_0u\right)^{-1+\varepsilon}\int_0^1 dv v^{-\varepsilon}(1-v)^{-\varepsilon}\,,
\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
u_0 &= \frac{P^2 - m_k^2}{P^2},\\
u &= \frac{2p_ip_j + 2p_jp_k}{2p_ip_k + 2p_ip_j + 2p_jp_k} = \frac{2p_ip_j + 2p_jp_k}{P^2 - m_k^2},\\
v &= \frac{2p_ip_j\left(2p_ip_k + m_k^2\right)}{2p_ip_k\left(2p_ip_j + 2p_jp_k\right)} = \frac{2p_ip_j\left(2p_ip_k + m_k^2\right)}{2p_ip_k\left(P^2 - m_k^2 - 2p_ip_k\right)}\,.
\end{aligned} \tag{4.16}$$

In the case where one mass vanishes ($m_j = 0$) and the other two are equal ($m_i = m_k = m$) we obtain

$$\begin{aligned}
d\phi_{\text{dipole}} &= \frac{(4\pi)^{\varepsilon-2}}{\Gamma(1-\varepsilon)}\left(P^2\right)^{1-\varepsilon}(r_0)^{2-2\varepsilon}\,2^{2\varepsilon-1} \times \\
&\quad \times \int_0^1 dr r^{1-2\varepsilon}(1-r)^{-\varepsilon+1/2}\left(1-r_0r\right)^{-1/2}\int_{-1}^1 ds\left(1-s^2\right)^{-\varepsilon}\,,
\end{aligned} \tag{4.17}$$



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with

$$r_0 = 1 - 4\frac{m^2}{P^2},$$
$$2p_ip_j = \frac{1}{2}P^2 r_0 r \left(1 - s\sqrt{\frac{r_0(1-r)}{1-r_0 r}}\right),$$
$$2p_jp_k = \frac{1}{2}P^2 r_0 r \left(1 + s\sqrt{\frac{r_0(1-r)}{1-r_0 r}}\right). \tag{4.18}$$

## 4.2 Phase space involving initial state particles

If initial state particles are involved, we obtain the following convolution:

$$d\phi(p_a + p_b \to K + p_k + p_i) = \int_0^1 dx d\phi(\tilde{p}_a + p_b \to K + \tilde{p}_k) d\phi_{\text{dipole}}, \tag{4.19}$$

with

$$\tilde{p}_a = x p_a. \tag{4.20}$$

In the case $m_a = m_i = 0$ we obtain

$$d\phi_{\text{dipole}} = \frac{(4\pi)^{\varepsilon-2}}{\Gamma(1-\varepsilon)}\left(-P^2\right)^{1-\varepsilon} x_0^{\varepsilon-1} x^{-1+\varepsilon} (1-x)^{1-2\varepsilon} (1-x_0 x)^{-1+\varepsilon} \int_0^1 dw w^{-\varepsilon}(1-w)^{-\varepsilon} \tag{4.21}$$

with $P = p_k + p_i - p_a$ and

$$x_0 = \frac{-P^2}{m_k^2 - P^2},$$
$$x = \frac{2p_ap_i + 2p_ap_k - 2p_ip_k}{2p_ap_i + 2p_ap_k},$$
$$w = \frac{2p_ap_i\left(2p_ip_k + m_k^2\right)}{2p_ip_k\left(2p_ap_i + 2p_ap_k\right)}. \tag{4.22}$$

Eq. (4.19) and eq. (4.21) are derived as follows: From the factorization of phase space we have

$$d\phi(p_a + p_b \to K + p_k + p_i) = \frac{1}{2\pi} d\phi(p_a + p_b \to K + Q) dm_Q^2 d\phi(Q \to p_i + p_k). \tag{4.23}$$

We then derive

$$d\phi(p_a + p_b \to K + Q) = x d\phi(\tilde{p}_a + p_b \to K + \tilde{p}_k). \tag{4.24}$$

This equation can be obtained by writing out the explicit expressions for the phase space measures in the rest frame of $p_a + p_b$ and $\tilde{p}_a + p_b$, respectively. These two frames are related by a boost. Further, since $\tilde{p}_a = x p_a$, the boost does not affect the transverse components. Next, one writes out the parameterization for $d\phi(Q \to p_i + p_k)$ in terms of solid angles of particle $p_k$, singles out one angle $\theta_1 = \angle(p_k, p_a)$ and replaces the integrations over $dm_Q^2 d\cos\theta_1$ by integrations over $dxdw$.

## 5. The dipole terms for final-state singularities with no initial-state partons

In this section we give the dipole factors corresponding to the case where all relevant partons are in the final state. Initial-state partons are considered in the next section. We distinguish the cases of (i) a massless emitter and massive spectator, (ii) a massive emitter and a massless spectator and (iii) a massive emitter and a massive spectator of equal mass. We follow closely the notation of Catani and Seymour [15].

The generic form of the dipole terms is given by

$$\mathcal{D}_{ij,k} = -\frac{1}{2p_i \cdot p_j} \langle 1, \ldots, (\tilde{ij}), \ldots, \tilde{k}, \ldots | \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} V_{ij,k} | 1, \ldots, (\tilde{ij}), \ldots, \tilde{k}, \ldots \rangle . \quad (5.1)$$

The explicit forms of the functions $V_{ij,k}$ are given below.

### 5.1 Massless final emitter, massive final spectator

We consider first the case of massless emitter (particle $i$) and a massive spectator (particle $k$), both in the final state. The variables $y$ and $z$ of Catani and Seymour are given by

$$y = \frac{2p_i p_j}{2p_i p_j + 2p_j p_k + 2p_i p_k}, \qquad z = \frac{2p_i p_k}{2p_i p_k + 2p_j p_k} \quad (5.2)$$

and related to the variables $u$ and $v$ as follows:

$$\begin{aligned} y &= u_0 v \frac{u(1-u)}{1-u_0 u}, \qquad u_0 = 1 - \frac{m_k^2}{(p_i+p_j+p_k)^2} \\ z(1-y) &= 1-u\,. \end{aligned} \quad (5.3)$$

In the collinear limit eq. (2.14) we have

$$y \to 0, \qquad z \to \hat{z} \quad (5.4)$$

and in the soft limit $p_j \to 0$

$$y \to 0\,, \qquad z \to 1\,. \quad (5.5)$$

The subtraction term for the splitting $g \to gg$ has in addition to match the soft limit $p_i \to 0$, corresponding to $y \to 0$, $z \to 0$.

As subtraction terms we use

$$\begin{aligned} \langle s | V_{q_i g_j,k} | s' \rangle &= 8\pi \mu^{2\varepsilon} \alpha_s C_F \delta_{ss'} \left[ \frac{2z(1-y)}{1-z(1-y)} + (1-\rho\varepsilon)(1-z(1-y)) \right], \\ \langle \mu | V_{q_i \bar{q}_j,k} | \nu \rangle &= 8\pi \mu^{2\varepsilon} \alpha_s T_R \left[ -g^{\mu\nu} - \frac{4}{2p_i p_j} S^{\mu\nu} \right], \\ \langle \mu | V_{g_i g_j,k} | \nu \rangle &= 16\pi \mu^{2\varepsilon} \alpha_s C_A \left[ -g^{\mu\nu} \left( \frac{z(1-y)}{1-z(1-y)} + \frac{(1-z)(1-y)}{1-(1-z)(1-y)} \right) + \right. \\ &\qquad \left. + (1-\rho\varepsilon) \frac{2}{2p_i p_j} S^{\mu\nu} \right], \end{aligned} \quad (5.6)$$

where the spin correlation tensor is given by

$$S^{\mu\nu} = (z(1-y)p_i^\mu - (1-z(1-y))p_j^\mu)(z(1-y)p_i^\nu - (1-z(1-y))p_j^\nu)\,. \tag{5.7}$$

The momenta are mapped as follows:

$$\begin{aligned}\tilde{p}_i &= ap_i + bp_j + cp_k\,,\\ \tilde{p}_k &= (1-a)p_i + (1-b)p_j + (1-c)p_k\,,\end{aligned} \tag{5.8}$$

with

$$\begin{aligned}c &= \frac{yu_0}{(2u-1-y(1-u))^2u_0 + 4u(1-u)y}\left(2u(1-u) - \frac{N}{\sqrt{1-v}}\right),\\ b &= \frac{1}{N}\left\{u + \frac{c}{yu_0}\left[-2yu - u_0\left((1-2u-u^2)y + 1 - 3u + 2u^2\right)\right]\right\},\\ a &= \frac{1}{N}\left\{1 - u - \frac{c}{yu_0}\left[2y(1-u) - u_0\left((1-u)y^2 + (1-u+u^2)y + u - 2u^2\right)\right]\right\},\\ N &= u^2 + (1-u)^2 + (1-u)y\,.\end{aligned} \tag{5.9}$$

This mapping satisfies momentum conservation

$$\tilde{p}_i + \tilde{p}_k = p_i + p_j + p_k\,, \tag{5.10}$$

and the on-shell conditions

$$\tilde{p}_i^2 = 0\,, \qquad \tilde{p}_k^2 = m_k^2\,. \tag{5.11}$$

In the limit $y \to 0$ we have

$$\lim_{y\to 0} \tilde{p}_i = p_i + p_j\,, \qquad \lim_{y\to 0} \tilde{p}_k = p_k\,. \tag{5.12}$$

In addition we have

$$S_{\mu\nu}\tilde{p}_i^\nu = 0\,, \tag{5.13}$$

e.g. $\tilde{p}_i$ is orthogonal to the spin correlation tensor. The integral over the spin correlation can be written as

$$\int d\phi_{\text{dipole}} \frac{-2}{(2p_ip_j)^2} S^{\mu\nu} = C_{21}\tilde{p}_i^\mu\tilde{p}_i^\nu + C_{22}\tilde{p}_k^\mu\tilde{p}_k^\nu + C_{23}\left(\tilde{p}_i^\mu\tilde{p}_k^\nu + \tilde{p}_k^\mu\tilde{p}_i^\nu\right) + C_{24}g^{\mu\nu}\,. \tag{5.14}$$

Using $S_{\mu\nu}\tilde{p}_i^\nu = 0$ and $\tilde{p}_i^2 = 0$ this reduces to

$$C_{21}\tilde{p}_i^\mu\tilde{p}_i^\nu - C_{24}\left(-g^{\mu\nu} + 2\frac{\tilde{p}_i^\mu\tilde{p}_k^\nu + \tilde{p}_k^\mu\tilde{p}_i^\nu}{2\tilde{p}_i\tilde{p}_k}\right). \tag{5.15}$$

Due to gauge invariance only the term $-C_{24}(-g^{\mu\nu})$ will give a non-vanishing contribution. $C_{24}$ is obtained by contracting with $g_{\mu\nu}$:

$$C_{24} = \frac{1}{2(1-\rho\varepsilon)} \int d\phi_{\text{dipole}} \frac{-2}{(2p_ip_j)^2} g_{\mu\nu}S^{\mu\nu}\,. \tag{5.16}$$

Integration of the subtraction terms yields:

$$\mathcal{V}_{q_i g_j,k} = \int d\phi_{\rm dipole} \frac{1}{2p_ip_j} V_{q_i g_j,k} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2 P^2}{(P^2-m_k^2)^2}\right)^{\varepsilon} \mathcal{V}_{qg}(u_0,\varepsilon)\,,$$

$$\begin{aligned}
\mathcal{V}_{qg}(u_0,\varepsilon) &= C_F\delta_{ss'}\int_0^1 du \int_0^1 dv u^{-2\varepsilon}(1-u)^{-2\varepsilon}(1-u_0u)^{\varepsilon} v^{-1-\varepsilon} \times \\
&\qquad \times (1-v)^{-\varepsilon}\left[2\frac{1-u}{u} + (1-\rho\varepsilon)u\right] \\
&= C_F\delta_{ss'}\frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\Gamma(2-2\varepsilon)\left[2\frac{\Gamma(-2\varepsilon)}{\Gamma(2-4\varepsilon)}{}_2F_1(-\varepsilon,-2\varepsilon;2-4\varepsilon;u_0) + \right. \\
&\quad \left. +(1-\rho\varepsilon)\frac{\Gamma(1-2\varepsilon)}{\Gamma(3-4\varepsilon)}{}_2F_1(-\varepsilon,2-2\varepsilon;3-4\varepsilon;u_0)\right] \\
&= C_F\delta_{ss'}\left[\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{17}{4} + \frac{1}{2}\rho - \frac{5}{6}\pi^2 + \frac{1}{2u_0} + \right. \\
&\qquad \left. + \frac{1}{2}\frac{(1-u_0)(1-3u_0)}{u_0^2}\ln(1-u_0) + 2{\rm Li}_2(u_0)\right] + O(\varepsilon),
\end{aligned}$$

$$\mathcal{V}_{q_i \bar{q}_j,k} = \int d\phi_{\rm dipole} \frac{1}{2p_ip_j} V_{q_i \bar{q}_j,k} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2 P^2}{(P^2-m_k^2)^2}\right)^{\varepsilon} \mathcal{V}_{q\bar{q}}(u_0,\varepsilon)\,,$$

$$\begin{aligned}
\mathcal{V}_{q\bar{q}}(u_0,\varepsilon) &= \int_0^1 du \int_0^1 dv u^{-2\varepsilon}(1-u)^{-2\varepsilon}(1-u_0u)^{\varepsilon} v^{-1-\varepsilon} \times \\
&\qquad \times (1-v)^{-\varepsilon} T_R\left[-g^{\mu\nu} - \frac{4}{2p_ip_j}S^{\mu\nu}\right] \\
&= T_R(-g^{\mu\nu})\frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left[\frac{\Gamma(1-2\varepsilon)^2}{\Gamma(2-4\varepsilon)}{}_2F_1(-\varepsilon,1-2\varepsilon,2-4\varepsilon,u_0)\right. \\
&\quad \left. -\frac{2}{1-\rho\varepsilon}\frac{\Gamma(2-2\varepsilon)^2}{\Gamma(4-4\varepsilon)}{}_2F_1(-\varepsilon,2-2\varepsilon,4-4\varepsilon,u_0)\right] + \text{gauge terms} \\
&= T_R(-g^{\mu\nu})\left[-\frac{2}{3\varepsilon} - \frac{13}{6} + \frac{1}{3}\rho + \frac{2}{3}\frac{(1-u_0)}{u_0^2} + \right. \\
&\qquad \left. + \frac{1}{3}\frac{(1-u_0)}{u_0^3}\left(2u_0^2 - u_0 + 2\right)\ln(1-u_0)\right] + \\
&\quad + \text{gauge terms} + O(\varepsilon)\,,
\end{aligned}$$

$$\mathcal{V}_{g_i g_j,k} = \int d\phi_{\rm dipole} \frac{1}{2p_ip_j} V_{g_i g_j,k} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2 P^2}{(P^2-m_k^2)^2}\right)^{\varepsilon} \mathcal{V}_{gg}(u_0,\varepsilon)\,,$$

$$\begin{aligned}
\mathcal{V}_{gg}(u_0,\varepsilon) &= 2C_A\int_0^1 du \int_0^1 dv u^{-2\varepsilon}(1-u)^{-2\varepsilon}(1-u_0u)^{\varepsilon} v^{-1-\varepsilon}(1-v)^{-\varepsilon} \times \\
&\qquad \times\left[-g^{\mu\nu}\left(\frac{1-u}{u} + \frac{u}{1-u} - \frac{u}{(1-u)}\frac{u_0 v}{(1-u_0u(1-v))}\right) + \right. \\
&\qquad \left. + (1-\rho\varepsilon)\frac{2}{2p_ip_j}S^{\mu\nu}\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2C_A(-g^{\mu\nu})\frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left[\frac{\Gamma(-2\varepsilon)\Gamma(2-2\varepsilon)}{\Gamma(2-4\varepsilon)}\Big({}_2F_1(-\varepsilon,-2\varepsilon,2-4\varepsilon,u_0)+\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad + {}_2F_1(-\varepsilon,2-2\varepsilon,2-4\varepsilon,u_0)\Big) +\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. +\frac{\Gamma(2-2\varepsilon)^2}{\Gamma(4-4\varepsilon)}{}_2F_1(-\varepsilon,2-2\varepsilon,4-4\varepsilon,u_0)\right] +\\
&\quad +2C_A(-g^{\mu\nu})(-u_0)\frac{\Gamma(1-\varepsilon)\Gamma(-2\varepsilon)}{\Gamma(-\varepsilon)}\times\\
&\quad \times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(n+1-\varepsilon)}{\Gamma(n+2-2\varepsilon)}\frac{\Gamma(m+n+2-2\varepsilon)}{\Gamma(m+n+2-4\varepsilon)}\frac{\Gamma(m-\varepsilon)}{m!}u_0^{n+m} + \text{gauge terms}\\
&= 2C_A(-g^{\mu\nu})\left[\frac{1}{\varepsilon^2}+\frac{11}{6}\frac{1}{\varepsilon}-\frac{5}{6}\pi^2+\frac{67}{12}-\frac{1}{3}\frac{(1-u_0)}{u_0^2}+2\,\mathrm{Li}_2(u_0)+\right.\\
&\qquad\qquad\qquad \left.-\frac{1}{6}\left(\frac{(1-u_0)}{u_0^3}(2-u_0)+11\frac{(1-u_0)}{u_0}\right)\ln(1-u_0)\right]\\
&\quad +\text{gauge terms}+O(\varepsilon)\,. \qquad (5.17)
\end{aligned}
$$

### 5.2 Massive final emitter, massless final spectator

We now consider the case of a massive emitter (particle $k$) and a massless spectator (particle $i$). It is sufficient to consider the case where a heavy quark (or antiquark) emits a gluon. We now have

$$
y = \frac{2p_jp_k}{2p_ip_j+2p_jp_k+2p_ip_k}\,, \qquad z = \frac{2p_ip_k}{2p_ip_k+2p_ip_j} \qquad (5.18)
$$

and

$$
\begin{aligned}
y &= u\left(1-\frac{(1-u)u_0}{1-u_0u}v\right), \qquad u_0 = 1-\frac{m_k^2}{(p_i+p_j+p_k)^2}\\
z(1-y) &= 1-u\,. \qquad (5.19)
\end{aligned}
$$

The momenta are mapped as follows:

$$
\begin{aligned}
\text{emitter}: \ \tilde{p}_k &= p_k+p_j-\frac{y}{1-y}p_i\,,\\
\text{spectator}: \ \tilde{p}_i &= \frac{1}{1-y}p_i\,. \qquad (5.20)
\end{aligned}
$$

Since there is no collinear singularity, we just have to match the part of the soft singularity which corresponds to this dipole factor:

$$
\langle s|V_{q_kg_j,i}|s'\rangle = 8\pi\mu^{2\varepsilon}\alpha_sC_F\delta_{ss'}\left[\frac{2z(1-y)}{1-z(1-y)}-2\left(\frac{1-u_0}{u_0}\right)\frac{1}{y}\right]. \qquad (5.21)
$$

With

$$
v_0 = \frac{u_0(1-u)}{1-u_0u} \qquad (5.22)
$$

we obtain for the integral over the dipole phase space

$$
\begin{aligned}
\mathcal{V}_{q_k g_j,i} &= \int d\phi_{\text{dipole}} \frac{1}{2p_j p_k} V_{q_k g_j,i} = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \left( \frac{4\pi\mu^2 P^2}{(P^2 - m_k)^2} \right)^{\varepsilon} \mathcal{V}_{Qg}(u_0, \varepsilon) \,, \\
\mathcal{V}_{Qg}(u_0,\varepsilon) &= C_F \delta_{ss'} \cdot 2 \int_0^1 du \int_0^1 dv u^{-2\varepsilon-1} (1-u)^{1-2\varepsilon} (1-u_0 u)^{\varepsilon-1} v^{-\varepsilon} (1-v)^{-\varepsilon} \times \\
&\qquad \times (1-v_0 v)^{-1} \left[ u_0 (1-u) - (1-u_0)(1-v_0 v)^{-1} \right] \\
&= C_F \delta_{ss'} \cdot 2\Gamma(1-\varepsilon) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(m-2\varepsilon)\Gamma(n+m+1-\varepsilon)}{\Gamma(n+m+2-4\varepsilon)} \times \\
&\quad \times \left( \frac{n+2-2\varepsilon}{n+m+2-4\varepsilon} u_0 - (n+1)(1-u_0) \right) \frac{u_0^{n+m}}{m!} \\
&= C_F \delta_{ss'} \left[ \frac{1}{\varepsilon} \left(1 + \ln(1-u_0)\right) + 4 + \ln(1-u_0) - \right. \\
&\qquad \left. - 4\mathrm{Li}_2(u_0) - \frac{1}{2} \ln^2(1-u_0) \right] + O(\varepsilon) \,. 
\end{aligned} \tag{5.23}
$$

Here we used

$$
\begin{aligned}
\sum_{i_1=2}^{\infty} \sum_{i_2=1}^{i_1-1} \frac{u_0^{i_1}}{i_1} \frac{1}{i_2} &= \mathrm{Li}_{1,1}(1, u_0) = H_{11}(u_0) = \frac{1}{2} \ln^2(1-u_0) \,, \\
\sum_{i_1=2}^{\infty} \sum_{i_2=1}^{i_1-1} u_0^{i_1} \frac{1}{i_2} &= \mathrm{Li}_{1,0}(1, u_0) = -\frac{u_0}{1-u_0} \ln(1-u_0) \,.
\end{aligned} \tag{5.24}
$$

### 5.3 Massive final emitter and massive final spectator of equal mass

We now consider the case of a massive emitter (particle $i$) and a massive spectator (particle $k$) of equal mass ($m_i = m_k = m$). It is sufficient to consider the case where a heavy quark (or antiquark) emits a gluon. Since there is no collinear singularity, we just have to match the part of the soft singularity which corresponds to this dipole factor:

$$
\begin{aligned}
\langle s | V_{q_i g_j,k} | s' \rangle &= \\
&= 8\pi\mu^{2\varepsilon} \alpha_s C_F \delta_{ss'} \sqrt{\frac{(2p_i p_j + 2p_j p_k + 2p_i p_k)^2 - (2m^2)^2}{(2p_i p_k)^2 - (2m^2)^2}} \left[ 2 \frac{2p_i p_k}{2p_i p_j + 2p_j p_k} - 2 \frac{m^2}{2p_i p_j} \right] \\
&= 8\pi\mu^{2\varepsilon} \alpha_s C_F \delta_{ss'} \frac{1}{r_0 r \sqrt{(1-r)(1-r_0 r)}} \left[ 2(1-r_0 r) - (1-r_0) - \frac{1-r_0}{1-s_0 s} \right] ,
\end{aligned} \tag{5.25}
$$

where

$$
\begin{aligned}
r &= \frac{2p_i p_j + 2p_j p_k}{2p_i p_j + 2p_j p_k + 2p_i p_k - 2m^2} \,, \\
s &= -\frac{1}{s_0} \frac{2p_i p_j - 2p_j p_k}{2p_i p_j + 2p_j p_k} \,,
\end{aligned}
$$

$$s_0 = \sqrt{\frac{r_0(1-r)}{1-r_0 r}}\,, \qquad r_0 = 1 - \frac{4m^2}{(p_i+p_j+p_k)^2}\,. \tag{5.26}$$

The singularity occurs for $r \to 0$. In this limit, the expression in the square root tends to 1. The inclusion of the square root term facilitates the analytic integration of the dipole term. The momenta are mapped as follows:

$$\begin{aligned}
\tilde{p}_i &= \frac{1}{2}P - \sqrt{\frac{r_0}{y_0^2-(1-r_0)}}\left(p_k - \frac{y_0}{2}P\right), \\
\tilde{p}_k &= \frac{1}{2}P + \sqrt{\frac{r_0}{y_0^2-(1-r_0)}}\left(p_k - \frac{y_0}{2}P\right),
\end{aligned} \tag{5.27}$$

with $P = p_i + p_j + p_k$ and

$$y_0 = \frac{2p_ip_k + 2p_jp_k + 2m^2}{P^2}\,. \tag{5.28}$$

This mapping satisfies the on-shell conditions $\tilde{p}_i^2 = \tilde{p}_k^2 = m^2$. In the soft limit $y_0$ tends to 1 and the correct asymptotic behaviour of the mapping is easily verified. Integration yields:

$$\mathcal{V}_{q_ig_j,k} = \int d\phi_{\text{dipole}} \frac{1}{2p_ip_j} V_{q_ig_j,k} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{P^2}\right)^{\varepsilon} \mathcal{V}_{QQ}(r_0,\varepsilon)\,,$$

$$\begin{aligned}
\mathcal{V}_{QQ}(r_0,\varepsilon) &= r_0^{-2\varepsilon} 2^{2\varepsilon} \int_0^1 dr r^{-2\varepsilon-1}(1-r)^{-\varepsilon}(1-r_0 r)^{-1} \int_{-1}^1 ds (1-s^2)^{-\varepsilon} \times \\
&\quad \times C_F \delta_{ss'} \left[ (2(1-r_0 r) - (1-r_0))(1-s_0 s)^{-1} - (1-r_0)(1-s_0 s)^{-2} \right] \\
&= C_F\delta_{ss'} \frac{1}{1-\varepsilon} 2^{2\varepsilon} r_0^{-2\varepsilon} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^i \left(1+(-1)^j\right) r_0^{k+j/2} \times \\
&\quad \times \frac{\Gamma(1+i+j)\Gamma(2-\varepsilon)}{\Gamma(2+i+j-\varepsilon)} \frac{\Gamma(i+\varepsilon)}{\Gamma(\varepsilon) i!} \frac{\Gamma(k+j/2)\Gamma(k-2\varepsilon)\Gamma(1+j/2-\varepsilon)}{\Gamma(j/2)\Gamma(k+j/2+1-3\varepsilon)k!} \times \\
&\quad \times \left[ 2 - (1-r_0)(2+j)\frac{j+2k}{j} \right] \\
&= C_F\delta_{ss'} \Bigg\{ \frac{1}{\varepsilon}\left(1 - \frac{1}{2}\frac{1+r_0}{\sqrt{r_0}} \ln\frac{1+\sqrt{r_0}}{1-\sqrt{r_0}}\right) - 2\ln r_0 + \ln(1-r_0) - \\
&\qquad - \ln^2\left(\frac{1+\sqrt{r_0}}{1-\sqrt{r_0}}\right) + \frac{1}{\sqrt{r_0}} \ln\left(\frac{1+\sqrt{r_0}}{1-\sqrt{r_0}}\right) - \\
&\qquad - \frac{1+r_0}{2\sqrt{r_0}} \Bigg( \mathrm{Li}_2\left(\sqrt{r_0}\right) - \mathrm{Li}_2\left(-\sqrt{r_0}\right) + \mathrm{Li}_2\left(\frac{1+\sqrt{r_0}}{2}\right) - \\
&\qquad\qquad - \mathrm{Li}_2\left(\frac{1-\sqrt{r_0}}{2}\right) + \mathrm{Li}_2\left(\frac{\sqrt{r_0}-1}{2\sqrt{r_0}}\right) - \\
&\qquad\qquad - \mathrm{Li}_2\left(\frac{\sqrt{r_0}-1}{\sqrt{r_0}}\right) + \mathrm{Li}_2\left(\frac{1}{1+\sqrt{r_0}}\right) -
\end{aligned}$$

$$- \mathrm{Li}_2\left(\frac{1-\sqrt{r_0}}{1+\sqrt{r_0}}\right) - 2\ln r_0 \ln\left(\frac{1+\sqrt{r_0}}{1-\sqrt{r_0}}\right) +$$
$$+ \ln 2 \ln\frac{\sqrt{r_0}}{1-\sqrt{r_0}} + \frac{1}{2}\ln^2 2 + \ln(1-\sqrt{r_0})\ln\left(\frac{1+\sqrt{r_0}}{\sqrt{r_0}}\right) +$$
$$+ \ln^2(1+\sqrt{r_0}) - \ln^2(1-\sqrt{r_0})\Bigg)\Bigg\} + O(\varepsilon)\,. \tag{5.29}$$

Here we used

$$\begin{aligned}
\sum_{i_1=2}^{\infty}\sum_{i_2=1}^{i_1-1}\frac{x_1^{i_1}}{i_1}\frac{x_2^{i_2}}{i_2} &= \mathrm{Li}_{1,1}(x_2,x_1) \\
&= \ln(1-x_1)\ln(1-x_2) + \mathrm{Li}_2\left(-\frac{x_2}{1-x_2}\right) - \mathrm{Li}_2\left(-\frac{x_2(1-x_1)}{1-x_2}\right) \\
&= -\frac{1}{2}\ln^2(1-x_1x_2) + \ln(x_2(1-x_1))\ln(1-x_1x_2) - \\
&\quad -\mathrm{Li}_2\left(\frac{1-x_2}{1-x_1x_2}\right) + \mathrm{Li}_2(1-x_2)\,, \\
\sum_{i_1=2}^{\infty}\sum_{i_2=1}^{i_1-1} x_1^{i_1}\frac{x_2^{i_2}}{i_2} &= \mathrm{Li}_{1,0}(x_2,x_1) = -\frac{x_1}{1-x_1}\ln(1-x_1x_2)\,, \\
\sum_{i_1=2}^{\infty}\sum_{i_2=1}^{i_1-1}\frac{x_1^{i_1}}{i_1}x_2^{i_2} &= \mathrm{Li}_{0,1}(x_2,x_1) = \frac{1}{1-x_2}\ln(1-x_1x_2) - \frac{x_2}{1-x_2}\ln(1-x_1)\,.
\end{aligned} \tag{5.30}$$

## 6. The dipole terms with initial-state partons

In this section we consider initial-state partons. We distinguish the cases of (i) a massless emitter in the initial state and massive spectator in the final state, (ii) a massive emitter in the final state and a massless spectator in the initial state. The generic form of the dipole terms for case (i) is given by

$$\mathcal{D}_k^{ai} = -\frac{1}{2p_a\cdot p_i}\frac{1}{x}\langle 1,\ldots,\tilde{(ai)},\ldots,\tilde{k},\ldots|\frac{\mathbf{T}_k\cdot\mathbf{T}_{ai}}{\mathbf{T}_{ai}^2}V_k^{ai}|1,\ldots,\tilde{(ai)},\ldots,\tilde{k},\ldots\rangle\,. \tag{6.1}$$

In the case (ii) we have

$$\mathcal{D}_{ki}^{a} = -\frac{1}{2p_i\cdot p_k}\frac{1}{x}\langle 1,\ldots,\tilde{(ki)},\ldots,\tilde{a},\ldots|\frac{\mathbf{T}_a\cdot\mathbf{T}_{ki}}{\mathbf{T}_{ki}^2}V_{ki}^{a}|1,\ldots,\tilde{(ki)},\ldots,\tilde{a},\ldots\rangle\,. \tag{6.2}$$

The explicit forms of the functions $V_k^{ai}$ and $V_{ki}^a$ are given below.

### 6.1 Massless initial emitter, massive final spectator

We now consider the case of a massless emitter (particle $a$) in the initial state and a massive spectator (particle $k$) in the final state. The variables $u$ and $x$ of Catani and Seymour are given by

$$u = \frac{2p_i p_a}{2p_i p_a + 2p_k p_a}\,, \qquad x = \frac{2p_i p_a + 2p_k p_a - 2p_i p_k}{2p_i p_a + 2p_k p_a}\,. \tag{6.3}$$

The variables $u$ and $w$ are related by

$$u = \frac{1-x}{1-x_0 x} w\,, \qquad x_0 = 1 - \frac{m_k^2}{m_k^2 - (p_k + p_i - p_a)^2}\,. \tag{6.4}$$

As subtraction terms we use

$$\begin{aligned}
\langle s|V_k^{q_a g_i}|s'\rangle &= 8\pi\mu^{2\varepsilon}\alpha_s C_F \delta_{ss'} \left[\frac{2}{1-x+u} - (1+x) - \rho\varepsilon(1-x)\right], \\
\langle s|V_k^{g_a \bar{q}_i}|s'\rangle &= 8\pi\mu^{2\varepsilon}\alpha_s T_R \delta_{ss'} \left[1 - \rho\varepsilon - 2x(1-x)\right], \\
\langle \mu|V_k^{q_a q_i}|\nu\rangle &= 8\pi\mu^{2\varepsilon}\alpha_s C_F \left[-g^{\mu\nu} x + 4\frac{(1-x)}{x}\frac{r(1-r)}{2p_i q} S^{\mu\nu}\right], \\
\langle \mu|V_k^{g_a g_i}|\nu\rangle &= 16\pi\mu^{2\varepsilon}\alpha_s C_A \left[-g^{\mu\nu}\left(\frac{1}{1-x+u} - 1 + x(1-x)\right) + \right. \\
&\qquad \left. +2(1-\rho\varepsilon)\frac{(1-x)}{x}\frac{r(1-r)}{2p_i q} S^{\mu\nu}\right],
\end{aligned} \tag{6.5}$$

with

$$S^{\mu\nu} = \left(\frac{1}{r}p_i^\mu - \frac{1}{1-r}q^\mu\right)\left(\frac{1}{r}p_i^\nu - \frac{1}{1-r}q^\nu\right). \tag{6.6}$$

Here $q$ is an arbitrary null vector not equal to $p_i$ and

$$r = \frac{2p_i p_a}{2p_i p_a + 2q p_a}\,. \tag{6.7}$$

The momenta are mapped as follows:

$$\begin{aligned}
\tilde{p}_a &= x p_a\,, \\
\tilde{p}_k &= p_k + p_i - (1-x) p_a\,.
\end{aligned} \tag{6.8}$$

This mapping satisfies momentum conservation

$$\tilde{p}_k - \tilde{p}_a = p_k + p_i - p_a\,, \tag{6.9}$$

and the on-shell conditions

$$\tilde{p}_a^2 = 0\,, \qquad \tilde{p}_k^2 = m_k^2\,. \tag{6.10}$$

The desired asymptotic behaviour

$$\lim_{u\to 0} \tilde{p}_a = x p_a\,, \qquad \lim_{u\to 0} \tilde{p}_k = p_k \tag{6.11}$$

is fulfilled. In addition we have

$$S_{\mu\nu}\tilde{p}_a^\nu = 0\,, \tag{6.12}$$

e.g. $\tilde{p}_a$ is orthogonal to the spin correlation tensor. We start with the integration of $V_k^{q_a g_i}$. We obtain

$$\begin{aligned}
\mathcal{V}_k^{q_a g_i} &= \int d\phi_{\text{dipole}}\frac{1}{2p_i p_a}V_k^{q_a g_i} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{m_k^2-P^2}\right)^\varepsilon \mathcal{V}^{qq}(x,x_0,\varepsilon),\\
\mathcal{V}^{qq}(x,x_0,\varepsilon) &= x^\varepsilon(1-x)^{-2\varepsilon}(1-x_0x)^\varepsilon\int_0^1 dww^{-\varepsilon-1}(1-w)^{-\varepsilon}\times\\
&\quad\times C_F\delta_{ss'}\left[\frac{2}{1-x+u}-(1+x)-\rho\varepsilon(1-x)\right]\\
&= C_F\delta_{ss'}x^\varepsilon(1-x)^{-2\varepsilon}(1-x_0x)^\varepsilon\frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\times\\
&\quad\times\left(\frac{2}{1-x}{}_2F_1\left(1,-\varepsilon;1-2\varepsilon;\frac{-1}{1-x_0x}\right)-(1+x)-\rho\varepsilon(1-x)\right).
\end{aligned} \tag{6.13}$$

$\mathcal{V}^{qq}$ is a distribution in $x$. In order to obtain expressions which are integrable at $x=1$ we rewrite

$$(1-x)^{-2\varepsilon-1} = (1-x)^{-2\varepsilon-1}\Big|_+ + \delta(1-x)\int_0^1 dy(1-y)^{-2\varepsilon-1}\,, \tag{6.14}$$

where

$$(1-x)^{-2\varepsilon-1}\Big|_+ = \frac{1}{1-x}\bigg|_+ - 2\varepsilon\,\frac{\ln(1-x)}{1-x}\bigg|_+ + O(\varepsilon^2)\,. \tag{6.15}$$

In order to expand the hypergeometric function we use the analytical continuation formula [29]

$$\begin{aligned}
{}_2F_1(a,b;c;-x) &= (x)^{-a}\,\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}{}_2F_1\left(a,1+a-c;1+a-b;-\frac{1}{x}\right)\\
&\quad+(x)^{-b}\,\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}{}_2F_1\left(b,1+b-c;1+b-a;-\frac{1}{x}\right),\\
&\qquad|\text{arg}x|<\pi.
\end{aligned} \tag{6.16}$$

We obtain

$$\begin{aligned}
\mathcal{V}^{qq}(x,x_0,\varepsilon) &= C_F\delta_{ss'}\Bigg[-\frac{1}{\varepsilon}\left(\frac{2}{1-x}\bigg|_+-(1+x)\right)+\rho(1-x)+\\
&\quad+\delta(1-x)\bigg(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln(2-x_0)+\frac{\pi^2}{6}+2\ln(1-x_0)\ln(2-x_0)+\\
&\qquad+2\text{Li}_2(x_0-1)-\frac{1}{2}\ln^2(2-x_0)\bigg)-\\
&\quad-2\left(\ln x+\ln(2-x_0x)\right)\left(\frac{1}{1-x}\bigg|_+\right)+4\,\frac{\ln(1-x)}{1-x}\bigg|_++\\
&\quad+(1+x)\left(\ln x-2\ln(1-x)+\ln(1-x_0x)\right)\Bigg]+O(\varepsilon)\,.
\end{aligned} \tag{6.17}$$

For the integration of the other dipole terms we proceed in a similar way and we obtain

$$\mathcal{V}_k^{g_a\bar{q}_i} = \int d\phi_{\rm dipole}\frac{1}{2p_ip_a}\frac{n_s(q)}{n_s(g)}V_k^{g_a\bar{q}_i} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{m_k^2-P^2}\right)^{\varepsilon}\mathcal{V}^{gq}(x,x_0,\varepsilon),$$

$$\begin{aligned}
\mathcal{V}^{gq}(x,x_0,\varepsilon) &= x^{\varepsilon}(1-x)^{-2\varepsilon}(1-x_0x)^{\varepsilon}\int_0^1 dw w^{-\varepsilon-1}(1-w)^{-\varepsilon}T_R\delta_{ss'}\frac{1-\rho\varepsilon-2x(1-x)}{1-\rho\varepsilon}\\
&= T_R\delta_{ss'}x^{\varepsilon}(1-x)^{-2\varepsilon}(1-x_0x)^{\varepsilon}\frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\frac{1-\rho\varepsilon-2x(1-x)}{1-\rho\varepsilon}\\
&= T_R\delta_{ss'}\Bigg[-\frac{1}{\varepsilon}\left(x^2+(1-x)^2\right)+2\rho x(1-x)-\\
&\qquad -\left(x^2+(1-x)^2\right)\Big(\ln x-2\ln(1-x)+\ln(1-x_0x)\Big)\Bigg]+O(\varepsilon)\,,
\end{aligned}$$

$$\mathcal{V}_k^{g_aq_i} = \int d\phi_{\rm dipole}\frac{1}{2p_ip_a}\frac{n_s(g)}{n_s(q)}V_k^{g_aq_i} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{m_k^2-P^2}\right)^{\varepsilon}\mathcal{V}^{qg}(x,x_0,\varepsilon),$$

$$\begin{aligned}
\mathcal{V}^{qg}(x,x_0,\varepsilon) &= x^{\varepsilon}(1-x)^{-2\varepsilon}(1-x_0x)^{\varepsilon}\int_0^1 dw w^{-\varepsilon-1}(1-w)^{-\varepsilon}\times\\
&\quad\times C_F\left[-g^{\mu\nu}x+4\frac{1-x}{x}\frac{r(1-r)}{2p_iq}S^{\mu\nu}\right](1-\rho\varepsilon)\\
&= C_F\left(-g^{\mu\nu}\right)\frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}x^{\varepsilon}(1-x)^{-2\varepsilon}(1-x_0x)^{\varepsilon}\times\\
&\quad\times\left(x+\frac{2}{1-\rho\varepsilon}\frac{1-x}{x}\right)(1-\rho\varepsilon)+\text{gauge terms}\\
&= C_F\left(-g^{\mu\nu}\right)\Bigg[-\frac{1}{\varepsilon}\left(x+2\frac{1-x}{x}\right)+\rho x-\\
&\qquad -\left(x+2\frac{1-x}{x}\right)\Big(\ln x-2\ln(1-x)+\ln(1-x_0x)\Big)\Bigg]+\\
&\quad+\text{gauge terms}+O(\varepsilon)\,,
\end{aligned}$$

$$\mathcal{V}_k^{g_ag_i} = \int d\phi_{\rm dipole}\frac{1}{2p_ip_a}V_k^{g_ag_i} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{m_k^2-P^2}\right)^{\varepsilon}\mathcal{V}^{gg}(x,x_0,\varepsilon),$$

$$\begin{aligned}
\mathcal{V}^{gg}(x,x_0,\varepsilon) &= x^{\varepsilon}(1-x)^{-2\varepsilon}(1-x_0x)^{\varepsilon}\int_0^1 dw w^{-\varepsilon-1}(1-w)^{-\varepsilon}\times\\
&\quad\times C_A\left[-g^{\mu\nu}\left(\frac{1}{1-x+u}-1+x-x^2\right)+2(1-\rho\varepsilon)\frac{1-x}{x}\frac{r(1-r)}{2p_iq}S^{\mu\nu}\right]\\
&= 2C_A\left(-g^{\mu\nu}\right)x^{\varepsilon}(1-x)^{-2\varepsilon}(1-x_0x)^{\varepsilon}\frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\times\\
&\quad\times\left[\frac{1}{1-x}{}_2F_1\left(1,-\varepsilon;1-2\varepsilon;\frac{-1}{1-x_0x}\right)-1+x-x^2+\frac{1-x}{x}\right]+\\
&\quad+\text{gauge terms}
\end{aligned}$$

$$
\begin{aligned}
= 2C_A\,(-g^{\mu\nu}) \Bigg[ & -\frac{1}{\varepsilon}\,\frac{1}{1-x}\bigg|_+ - (\ln x + \ln(2-x_0 x))\left(\frac{1}{1-x}\bigg|_+\right) + \\
& + 2\,\frac{\ln(1-x)}{1-x}\bigg|_+ + \left(\frac{1}{\varepsilon} + \ln x - 2\ln(1-x) + \ln(1-x_0 x)\right) \times \\
& \times \left(1 - x + x^2 - \frac{1-x}{x}\right) + \\
& + \delta(1-x)\left(\frac{1}{2\varepsilon^2} + \frac{\pi^2}{12} + \frac{1}{2\varepsilon}\ln(2-x_0) + \right. \\
& \qquad + \ln(1-x_0)\ln(2-x_0) + \mathrm{Li}_2(x_0-1) - \\
& \qquad \left. - \frac{1}{4}\ln^2(2-x_0)\right)\Bigg] + \text{gauge terms} + O(\varepsilon)\,.
\end{aligned} \tag{6.18}
$$

Here $n_s(q) = 2$ denotes the polarizations of a fermion and $n_s(g) = 2(1-\rho\varepsilon)$ denotes the polariztions of a gluon. Note that the dependece on the momentum $q$ in $\mathcal{V}_k^{q_a q_i}$ and $\mathcal{V}_k^{g_a g_i}$ dropped out after integration.

### 6.2 Massive final emitter, massless initial spectator

We now consider the case of a massive emitter (particle $k$) in the final state and a massless spectator (particle $a$) in the initial state. It is sufficient to consider the case where a heavy quark (or antiquark) emits a gluon. The variables $z$ and $x$ of Catani and Seymour are given by

$$
z = \frac{2p_k p_a}{2p_i p_a + 2p_k p_a}\,, \qquad x = \frac{2p_i p_a + 2p_k p_a - 2p_i p_k}{2p_i p_a + 2p_k p_a}\,. \tag{6.19}
$$

The variables $z$ and $w$ are related by

$$
z = 1 - \frac{1-x}{1-x_0 x}\,, \qquad x_0 = 1 - \frac{m_k^2}{m_k^2 - (p_k + p_i - p_a)^2}\,. \tag{6.20}
$$

Since there is no collinear singularity, we just have to match the part of the soft singularity which corresponds to this dipole factor:

$$
\langle s|V^a_{q_k g_i}|s'\rangle = 8\pi\mu^{2\varepsilon}\alpha_s C_F \delta_{ss'}\left[\frac{2}{1-z+(1-x)} - 2(1-x_0)\frac{x^2}{1-x}\right]. \tag{6.21}
$$

We use the same mapping for the momenta as in the previous section:

$$
\begin{aligned}
\tilde{p}_a &= x p_a\,, \\
\tilde{p}_k &= p_k + p_i - (1-x)p_a\,.
\end{aligned} \tag{6.22}
$$

This mapping fulfills the asymptotic behaviour

$$
\lim_{2p_k p_i \to 0} x = 1\,, \qquad \lim_{2p_k p_i \to 0} \tilde{p}_k = p_k + p_i\,, \qquad \lim_{2p_k p_i \to 0} \tilde{p}_a = p_a\,. \tag{6.23}
$$

Integration yields

$$\mathcal{V}^a_{q_k g_i} = \int d\phi_{\rm dipole} \frac{1}{2p_i p_k} V^a_{q_k g_i} = \frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu^2}{m_k^2 - P^2}\right)^{\varepsilon} \mathcal{V}_{Qg}(x, x_0, \varepsilon)\,,$$

$$\begin{aligned}
\mathcal{V}_{Qg}(x,x_0,\varepsilon) &= x^{\varepsilon}(1-x)^{-2\varepsilon}(1-x_0 x)^{\varepsilon-1}\int_0^1 dw w^{-\varepsilon}(1-w)^{-\varepsilon}\times \\
&\quad \times C_F\left[\frac{2}{1-z+(1-x)} - 2(1-x_0)\frac{x^2}{1-x}\right] \\
&= C_F\delta_{ss'} x^{\varepsilon}(1-x)^{-2\varepsilon}(1-x_0 x)^{\varepsilon-1}\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}\times \\
&\quad \times\left(\frac{2}{1-x}{}_2F_1\left(1, 1-\varepsilon; 2-2\varepsilon; \frac{-1}{1-x_0 x}\right) - 2(1-x_0)\frac{x^2}{1-x}\right) \\
&= C_F\delta_{ss'}\left[\delta(1-x)\left(\frac{1}{\varepsilon}\Big(1+\ln(1-x_0)-\ln(2-x_0)\Big) + 2 - \frac{\pi^2}{3} + \right.\right. \\
&\qquad\qquad + \ln(1-x_0) + \frac{1}{2}\ln^2(1-x_0) + \frac{1}{2}\ln^2(2-x_0) - \\
&\qquad\qquad \left. - 2\ln(1-x_0)\ln(2-x_0) - 2\mathrm{Li}_2(x_0-1)\right) + \\
&\quad \left.+2\left(\ln(2-x_0 x) - \ln(1-x_0 x) - \frac{(1-x_0)x^2}{1-x_0 x}\right)\left(\frac{1}{1-x}\bigg|_+\right)\right] + O(\varepsilon)\,.
\end{aligned} \qquad (6.24)$$

## 7. Conclusions

In this paper we have extended the dipole formalism to processes involving heavy fermions. We gave the explicit subtraction terms, together with a mapping of the momenta from the $(n+1)$-parton configuration to the $n$-parton configuration. We evaluated the integrals of the subtraction terms over the dipole phase space to order $O(\varepsilon^0)$. These ingredients are sufficient to set up numerical NLO programs based on the dipole formalism for processes involving heavy fermions. An application to single-top production is in preparation [30]. Furthermore we used a systematic approach for the integration and subsequent expansion in $\varepsilon$. This method allows us in principle to obtain higher orders in $\varepsilon$. Our results are naturally expressed in terms of harmonic polylogarithms. We are therefore confident that the techniques developed in this paper will prove useful also in the cancellation of infrared singularities at next-to-next-to-leading order (NNLO).

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## Erratum

Equation (5.29) in section 5.3 is not correct. It should read:

$$\mathcal{V}_{q_i g_j,k} = \int d\phi_{dipole} \frac{1}{2p_i p_j} V_{q_i g_j,k} = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu^2}{P^2}\right)^{\varepsilon} \mathcal{V}_{QQ}(r_0,\varepsilon),$$

$$\begin{aligned}
\mathcal{V}_{QQ}(r_0,\varepsilon) &= r_0^{-2\varepsilon} 2^{2\varepsilon} \int\limits_0^1 dr r^{-2\varepsilon-1}(1-r)^{-\varepsilon}(1-r_0 r)^{-1} \int\limits_{-1}^1 ds (1-s^2)^{-\varepsilon} \times \\
&\quad \times C_F \delta_{ss'} \left[\left(2(1-r_0 r) - (1-r_0)\right)(1-s_0 s)^{-1} - (1-r_0)(1-s_0 s)^{-2}\right] \\
&= C_F \delta_{ss'} \frac{1}{1-\varepsilon} 2^{2\varepsilon} r_0^{-2\varepsilon} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^i \left(1+(-1)^j\right) r_0^{k+j/2} \times \\
&\quad \times \frac{\Gamma(1+i+j)\Gamma(2-\varepsilon)}{\Gamma(2+i+j-\varepsilon)} \frac{\Gamma(i+\varepsilon)}{\Gamma(\varepsilon) i!} \frac{\Gamma(k+j/2)\Gamma(k-2\varepsilon)\Gamma(1+j/2-\varepsilon)}{\Gamma(j/2)\Gamma(k+j/2+1-3\varepsilon)k!} \times \\
&\quad \times \left[2 - (1-r_0)(2+j)\frac{j+2k}{j}\right] \\
&= C_F \delta_{ss'} \left\{ \frac{1}{\varepsilon}\left(1 - \frac{1}{2}\frac{1+r_0}{\sqrt{r_0}} \ln\frac{1+\sqrt{r_0}}{1-\sqrt{r_0}}\right) - 2\ln r_0 - \right. \\
&\qquad - \ln^2\left(\frac{1+\sqrt{r_0}}{1-\sqrt{r_0}}\right) + \frac{1}{\sqrt{r_0}} \ln\left(\frac{1+\sqrt{r_0}}{1-\sqrt{r_0}}\right) - \\
&\qquad - \frac{1+r_0}{2\sqrt{r_0}} \left( \mathrm{Li}_2\left(\sqrt{r_0}\right) - \mathrm{Li}_2\left(-\sqrt{r_0}\right) + 2\,\mathrm{Li}_2\left(\frac{1+\sqrt{r_0}}{2}\right) - \right. \\
&\qquad\qquad - 2\,\mathrm{Li}_2\left(\frac{1-\sqrt{r_0}}{2}\right) + \mathrm{Li}_2\left(\frac{\sqrt{r_0}-1}{2\sqrt{r_0}}\right) - \\
&\qquad\qquad - \mathrm{Li}_2\left(\frac{\sqrt{r_0}-1}{\sqrt{r_0}}\right) + \mathrm{Li}_2\left(\frac{1}{1+\sqrt{r_0}}\right) - \\
&\qquad\qquad - \mathrm{Li}_2\left(\frac{1-\sqrt{r_0}}{1+\sqrt{r_0}}\right) - 2\ln r_0 \ln\left(\frac{1+\sqrt{r_0}}{1-\sqrt{r_0}}\right) + \\
&\qquad\qquad + \ln 2 \ln\frac{\sqrt{r_0}}{1+\sqrt{r_0}} + \frac{1}{2}\ln^2 2 \\
&\qquad\qquad + \ln(1-\sqrt{r_0}) \ln\left(\frac{1+\sqrt{r_0}}{\sqrt{r_0}}\right) \\
&\qquad\qquad \left.\left. + \frac{1}{2}\ln^2(1+\sqrt{r_0}) - \frac{1}{2}\ln^2(1-\sqrt{r_0}) \right) \right\} \\
&\quad + O(\varepsilon).
\end{aligned}$$



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